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Structure of the Set of Annihilators

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We investigate the right annihilator lattice of a *-ring and ask whether it is orthomodular with respect to a naturally given involution. In particular, we introduce a new class of *-rings with orthomodular right annihilator lattice.

INTRODUCTION

It is well known that the projection set of a Baer*-ring (Berberian, 1972) forms a complete orthomodular lattice.

There are many *-rings which only have a few projections, but many right (respectively left) annihilators. Since a right annihilator forms a substitute for a projection in a natural way, it makes sense to investigate the poset of the right annihilators instead of the projection poset.

We shall start our considerations with the lattice of regular subsets corresponding to an orthogonality relation (Kalmbach, 1983) on a given set. Applying this general concept, we consider the right annihilator set of a *-ring and, more generally, of a *-semigroup.

In all cases, the main question dealt with is whether the obtained lattices are orthomodular.

In particular, we shall prove orthomodularity for a class of *-rings in which the additive structure and the structure of the right annihilator set are closely connected.

1. SYMMETRIC RELATIONS AND ORTHOMODULARITY

Let $(S, \perp)$ be a set S with a symmetric relation $\perp$ on it. Then ${}^{\perp}: 2^S \to 2^S$ with $M^{\perp} = \{s \in S \mid (\forall t \in M)(s \perp t)\}$ is a closure operator, and it is well known that the set $\mathscr{R} = \{M \subseteq S \mid M^{\perp\perp} = M\}$ of all ${}^{\perp}$-regular (${}^{\perp}$-closed) subsets of S

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forms a complete bounded lattice with respect to the restriction of the inclusion relation to $\mathscr{R}$. Here S is the greatest, $S^{\perp}$ the least element of $\mathscr{R}$. The infimum and the supremum of a family $(R_i)_{i\in I}$ are given, respectively, as follows:

$$\bigwedge_i R_i = \bigcap_i R_i, \qquad \bigvee_i R_i = \left(\bigcap_i R_i^{\perp}\right)^{\perp} = \left(\bigcup_i R_i\right)^{\perp\perp} \tag{1.1}$$

Moreover, $\perp$ is an *involution* on $\mathscr{R}$, i.e., for all $Q, R\in\mathscr{R}$,

$$R^{\perp\perp} = R \tag{1.2}$$

$$Q\leq R \quad \text{implies} \quad R^{\perp}\leq Q^{\perp} \tag{1.3}$$

In the theory of quantum structures the above relation $\perp$ (mostly requiring some additional conditions) is said to be an *orthogonality relation* [for an extensive survey, see Dietz (1984) and Kalmbach (1983)].

An algebra $(P;\ \vee,\ \wedge,\ {}^{\perp}, 0, 1)$ is called an *ortholattice* if $(P;\ \vee,\ \wedge)$ is a lattice with least element 0 and greatest element 1, and $\perp$ is an involution, with

$$1 = p\vee p^{\perp} \quad (\text{hence } 0 = p\wedge p^{\perp}) \qquad \text{for all } p\in P \tag{1.4}$$

It is said to be an *orthomodular lattice* if additionally, for all $p, q\in P$,

$$p\leq q \quad \text{implies} \quad q = p\vee(q\wedge p^{\perp}) \tag{1.5}$$

Many authors have investigated the assumptions under which an orthogonality relation (with additional conditions) induces an orthomodular lattice of ${}^{\perp}$-regular sets. Let us give conditions equivalent to orthomodularity for the general case. The first (b) (see below) was found by Finch (1970) and the second one (c) can be found in Dietz (1984, Section 1, Lemma 3). By an *orthogonal set* we mean a set $M\subseteq S$ with $s\perp t$ or $s=t$ for all $s, t\in M$, and $M\perp N$ $(M, N\subseteq S)$ means $s\perp t$ for all $s\in M$ and $t\in N$.

Proposition 1.1. Let $(S, \perp)$ be as above. Then:

(i) $(\mathscr{R};\ \wedge,\ \vee,\ {}^{\perp}, S^{\perp}, S)$ is an ortholattice iff $S^{\perp} = \{s\in S \mid s\perp s\}$.

(ii) If $S^{\perp} = \{s\in S\mid s\perp s\}$, the following conditions are equivalent:

(a) $\mathscr{R}$ is an orthomodular lattice.

(b) For each $R\in\mathscr{R}$ and every maximal orthogonal subset M of R, $M^{\perp\perp} = R$.

(c) For each orthogonal set M and each $s\in S$ there exists an orthogonal set M' with $M\subseteq M'$ and $(M\cup\{s\})^{\perp\perp} = (M')^{\perp\perp}$.

(d) For all $M, N\subseteq S$ and $s, t\in S$, $s\perp N\perp M\perp t$ and $(M\cup N)^{\perp\perp} = S$ implies $s\perp t$.

Proof. Since (i) is obvious, we only show (ii).

(a)$\Rightarrow$(b). Let $\mathscr{R}$ be orthomodular, let $R \in \mathscr{R}$, and let M be a maximal orthogonal subset of R. Then $M^{\perp\perp} \subseteq R$.

Assuming $M^{\perp\perp}) \subsetneq R$, by orthomodularity, one obtains $R \cap M^{\perp} \supsetneq S^{\perp}$. For any $s \in (R \cap M^{\perp}) \setminus S^{\perp}$, the set $M \cup \{s\}$ is an orthogonal subset of R, which contains M. This contradicts the maximality of M.

(b)$\Rightarrow$(c). Let $M \subseteq S$ be an orthogonal set, let $s \in S$, and assume that (b) is valid. By Zorn's lemma we can choose a maximal orthogonal subset M' of $(M \cup \{s\})^{\perp\perp}$ containing M. Now $(M \cup \{s\})^{\perp\perp} = (M')^{\perp\perp}$ follows from (b).

(c)$\Rightarrow$(a). Assume (c) and that $\mathscr{R}$ is not orthomodular. Then it is easy to see that there exist $R_1, R_2 \in \mathscr{R}$ with $R_1 \subsetneq R_2$, but $R_2 \cap R_1{}^{\perp} = S^{\perp}$.

Choose a maximal orthogonal subset M of R_1. If $M^{\perp\perp} = R_1$, let $s \in R_1 \setminus M^{\perp\perp}$. If $M^{\perp\perp} = R_1$, then M is also a maximal orthogonal subset of R_2. In this case let $s \in R_2 \setminus M^{\perp\perp}$.

Now it is easy to see that M cannot be extended to an orthogonal set M' with $(M \cup \{s\})^{\perp\perp} = (M')^{\perp\perp}$, which contradicts (c).

Finally, we show the equivalence of (a) and (d).

(d)$\Rightarrow$(a). Assume (d) and that $\mathscr{R}$ is not orthomodular. Then again choose $R_1, R_2 \in \mathscr{R}$ with $R_1 \subsetneq R_2$, but $R_2 \cap R_1{}^{\perp} = S^{\perp}$. For $M = R_1$ and $N = R_2^{\perp}$, $M \perp N$ and $(M \cup N)^{\perp\perp} = S$. By (d), $M^{\perp} \perp N^{\perp}$, which implies $R_1^{\perp} \subseteq R_2^{\perp}$, and this contradicts $R_1 \neq R_2$.

(a)$\Rightarrow$(d). Assume orthomodularity of $\mathscr{R}$ and let $M, N \subseteq S$ and $s, t \in S$ with $s \perp N \perp M \perp t$ and $(M \cup N)^{\perp\perp} = S$, but $s \not\perp t$. Then

$$M^{\perp} \supseteq (M \cup \{s\})^{\perp} \supseteq N^{\perp\perp}$$

and since $t \in M^{\perp}$, but $s \perp t$, $M^{\perp} \neq N^{\perp\perp}$. By orthomodularity, $M^{\perp} \cap N^{\perp} \supsetneq S^{\perp}$, which contradicts $(M^{\perp} \cap N^{\perp})^{\perp} = (M \cup N)^{\perp\perp} = S$. ∎

2. ON THE RIGHT ANNIHILATOR LATTICE OF A *-SEMIGROUP

A **-semigroup* $(S; \cdot, *)$ is a semigroup $(S; \cdot)$ with *involution*, i.e., a unary operation with

$$s^{**} = s \tag{2.1}$$

$$(s \cdot t)^* = t^* \cdot s^* \tag{2.2}$$

for all $s, t \in S$.

Assuming that S has a zero element 0, for $M \subseteq S$, let

$$\mathbf{l(M)} = \{s \in S \mid (\forall t \in M)(st = 0)\}$$

be the left annihilator and **r(M)** $=\{s\in S\mid(\forall t\in M)(ts=0)\}$ be the right annihilator of M. Investigating the structure of the set of right (respectively left) annihilators, we can apply the results of Section 1, namely $s\perp t$ iff $s^*t=0$ (respectively $s\perp' t$ iff $st^*=0$) defines a symmetric relation $\perp$ (respectively $\perp'$) on S. The following statements are easy to show and are left to the reader.

Lemma 2.1. Let S be a *-semigroup with 0. Then:

(i) $M^{\perp}=r(M^*)=l(M)^*$ [respectively $M^{\perp'}=l(M^*)=r(M)^*$] for all $M\subseteq S$.
(ii) $M^{\perp\perp}=rl(M)$ [respectively $M^{\perp'\perp'}=lr(M)$] for all $M\subseteq S$.
(iii) $\mathcal{R}=\{M^{\perp\perp}=M\mid M\subseteq S\}$ (respectively $\{M^{\perp'\perp'}=M\mid M\subseteq S\}$) is the set of all right (respectively left) annihilators. ■

Since * induces a correspondence between right and left annihilators, it suffices to study the right annihilator set.

The results in Section 1 imply the following result.

Proposition 2.2. If S is a *-semigroup with 0 and with *proper* involution, i.e., $s^*s=0$ implies $s=0$, then (a)–(d) of Proposition 1.1 are equivalent.

Example 2.3. Let $(P;\leq,0,1,i)$ be a partially ordered set with a least element 0, a greatest element 1, and an involution i, and let A be the set of all antitone maps on P. Further, let S be the set of all $s\in A$ for which there exists an element t of A such that (s,t) forms a Galois connection, i.e., $s\circ t(q)\geq q$, $t\circ s(q)\geq q$ for all $q\in P$. Since t is uniquely determined by s, which an easy computation shows, the notion $s^*=t$ makes sense. Note that the double (s_p, s_p^*) with $(q\in P)$

$$s_p(q)=\begin{cases}1 & \text{for} \quad q\leq p\\ 0 & \text{else}\end{cases} \tag{2.3}$$

$$s_p^*(q)=\begin{cases}1 & \text{for} \quad q=0\\ p & \text{else}\end{cases} \tag{2.4}$$

for all $p\in P$ forms a Galois connection with $s_p^*(1)=p$. An easy calculation shows that $(S_p,\cdot,*)$ with

$$s\cdot t=s\circ i\circ t \tag{2.5}$$

forms a *-semigroup with $0=s_1$. Note that i is a multiplicative unit of S.

Moreover,

$$r(M)=\{s\in S\mid(\forall t\in M)(s(1)\geq i\circ t^*(1))\}$$

for each subset M of S.

For $p \in P$, let $R_p = \{s \in S | s(1) \geq i(p)\}$. Then, taking into consideration that $s_p^*(1) = p$, $R_{i(p)} = R_p^\perp$ for all $p \in P$. This shows that $R_p \in \mathscr{R}$ for all $p \in P$. Moreover, $p \to R_p$ is an order-isomorphic embedding of P into $\mathscr{R}$, and one has $R_0 = \{s_1\}$ and $R_1 = S$. $\mathscr{R}$ is the Dedekind–MacNeille completion of $\{R_p | p \in P\}$. By Proposition 1.1(i), $\mathscr{R}$ is an ortholattice iff * is a proper involution. ■

An immediate consequence of the considerations in Example 2.3 is the following representation result (cf. also Blyth and Janowitz, 1972).

Proposition 2.4. Each complete lattice P with involution is isomorphic to the right annihilator lattice of a *-semigroup S with 0. If P is an ortholattice, the involution can be chosen to be proper.

A sufficiently well-investigated class of *-semigroups is the class of Foulis semigroups. We mention only that each orthomodular lattice can be represented by the projection lattice of a Foulis semigroup and refer the reader to Blyth and Janowitz (1972) and Foulis (1960) (an additional remark: Foulis, who introduced this class of *-semigroups, called them Baer*-semigroups).

3. A NEW CLASS OF *-RINGS

Let $(S; +, \cdot, {}^*)$ be a *-ring, i.e., the multiplicative semigroup of S forms a *-semigroup, and, for all $s, t \in S$,

$$(s+t)^* = s^* + t^* \tag{3.1}$$

The aim of this section is to investigate the order structure of the right annihilator set of such rings. Unfortunately, even in the commutative case the right annihilator lattice can have a very unsatisfactory structure, as the following example shows.

Example 3.1. Let S be the residue class ring modulo 4 with the identical involution *. Then $\mathscr{R} = \{\{0\}, S, \{0, 2\}\}$ fails to be an ortholattice; one has $\{0, 2\}^\perp = \{0, 2\}$. ■

Getting *-rings with nicer properties, we introduce the following concept:

Definition 3.2. A *-ring S with proper involution is said to be a ring with *annihilator addition property* (**AAP**-ring) if, for all $R \in \mathscr{R}$,

$$S = R + R^\perp \tag{3.2}$$

Theorem 3.3. Let S be an AAP-ring. Then $\mathscr{R}$ is a complete orthomodular lattice.

Proof. Let R_1, $R_2 \in \mathscr{R}$ and $R_1 \subseteq R_2$. Then for each $s \in R_2$ choose elements $Q(s)$ in R_1 and $P(s)$ in $R_1^{\perp}$ with $s = P(s) + Q(s)$ and let $X = \{P(s) \mid s \in R_2\}$ and $R_3 = rl(X)$.

We have $R_3 \subseteq R_1^{\perp}$, hence $R_3 \perp R_1$.

Moreover, for each $x \in X$, there exists an $s \in R_2$ with $x = P(s)$. Therefore, $x = s - Q(s) \in R_2$, and we obtain $X \subseteq R_2$, hence $R_3 \subseteq R_2$.

Finally, $R_2 \subseteq R_1 + X \subseteq R_1 + R_3 \subseteq R_1 \vee R_3 \subseteq R_2$.

Let us summarize: For all R_1, $R_2 \in \mathscr{R}$ with $R_1 \subseteq R_2$, there exists an $R_3 \in \mathscr{R}$ with $R_1 \perp R_3$ and $R_2 = R_1 \vee R_3$. This is equivalent to the orthomodularity of R. ■

The following example shows that in general orthomodularity of $\mathscr{R}$ does not imply that the given *-ring is an AAP-ring.

Example 3.4. The ring $C(I)$ of all complex-valued functions on a closed real interval I with $f^*(x) = \overline{f(x)}$ for all $f \in C(I)$ forms a commutative *-ring with proper involution. Taking into consideration that each (right) annihilator of a set M of functions on I corresponds to the regular-closed set $\mathrm{cl}(\{x \in I \mid (\exists f \in M)(f(x) \neq 0)\})$ (Engelking, 1977), it is easily seen that $\mathscr{R}$ is a Boolean algebra and that S fails to be an AAP-ring. ■

The example also shows that a sub-*-ring S_0 of an AAP-ring S need not be an AAP-ring, namely $C(I)$ is contained in the AAP-ring of all complex-valued functions on I.

Nevertheless, if we require that S admits a multiplicative unit 1 and that S_0 is an ideal, then the annihilator addition property is preserved.

Theorem 3.5. If S is an AAP-ring with 1, then each *-symmetric ideal S_0 of S is an AAP-ring.

Proof. Let R_0 be a right annihilator in S_0 and $R = R_0^{\perp\perp} = rl(R_0)$. Then $R^{\perp} = R_0^{\perp} \supseteq R_0'$, where the prime denotes the involution in the right annihilator lattice of S_0.

Further, $S_0 = SS_0 = RS_0 + R^{\perp}S_0$, and, since right annihilators form right ideals, $RS_0 \subseteq R \cap S_0$ and $R^{\perp}S_0 \subseteq R^{\perp} \cap S_0$. Therefore, $S_0 \subseteq R \cap S_0 + R^{\perp} \cap S_0$.

Finally, $R \cap S_0 = R_0^{\perp\perp} \cap S_0 \subseteq (R_0')^{\perp} \cap S_0 \subseteq R_0$ and $R^{\perp} \cap S_0 = R_0^{\perp} \cap S_0 = R_0'$; hence $S_0 = R_0 + R_0'$, which completes the proof. ■

Question 1. Does the statement remain valid if S has no unit?

A *-ring S is said to be a Baer*-ring (respectively Rickart*-ring) if, for each $R \in \mathscr{R}$ [respectively "principal" right annihilator $R = r(\{s\})$ with $s \in S$], there exists a projection $p \in S$, i.e., an element p with $p^2 = p = p^*$, such that $R = pS$.

We remark that in the literature the concept of a Baer*-ring is used in different ways. We follow the concept of Berberian (1972).

A Baer*-ring S admits a multiplicative unit 1, and, if $p \in S$ is a projection, then also $1-p$ is a projection. Moreover, $(pS)^{\perp} = (1-p)S$. Therefore, Baer*-rings are APP-rings. This is not true for Rickart*-rings, as will be seen below. One of the best known examples of a Baer*-ring is the *-ring of bounded linear operators on a complex Hilbert space, where T^* is defined as the adjoint operator of a given operator T. It contains the *-symmetric ideal of all compact operators, which forms a Rickart*-algebra.

From Theorems 3.3 and 3.5 immediately follows:

Corollary 3.6. The right annihilator lattice of the *-ring of all compact operators is a complete orthomodular lattice.

Finally, let us show that, in the commutative case, AAP-rings do not form a new class of *-rings.

Theorem 3.7. A commutative *-ring S is an AAP-ring iff it is a Baer*-ring.

Proof. Let S be a commutative AAP-ring and $R \in \mathscr{R}$. Choose elements s in R and t in $R^{\perp}$ with $1 = s + t$. Then $1 = s^*s + t^*t$ and, therefore, $s^*s = (s^*s)^2$ and $t^*t = (t^*t)^2$. Now, $p = s^*s$ and $1 - p = t^*t$ are projections and, obviously, $p \in R$, $1 - p \in R^{\perp}$, $pS \subseteq R$, and $(1-p)S \subseteq R^{\perp}$. We show $pS = R$. If $u \in R \backslash pS$, then $u(1-p) \in R \cap R^{\perp}$, hence $u(1-p) = 0$. From this we obtain $u = up \in pS$, which contradicts our assumption.

This shows that S is a Baer*-ring. ■

It is well known that there exist commutative Rickart*-rings which fail to be a Baer*-ring, for example, the *-ring of all complex-valued functions on the Cantor discontinuum. These *-rings cannot be AAP-rings by Theorem 3.7.

Obviously, the product of AAP-rings is an AAP-ring, but we do not know the behavior of AAP-rings under homomorphisms.

Question 2. Is the homomorphic image of an AAP-ring an AAP-ring?

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